K-moduli space of del pezzo surface pairs

Joint work with Long Pan and Haoyu Wu

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- In 2017, Chi Li and K. Fujita discover the valuative criterion for K-stability, where many birational geometric tools can apply.
- In the recent years, Xu's school developed algebraic K-stability theory and use the theory to construct good moduli spaces for K-polystable (log) Fano varieties.

K-stability: definition

Recall a log Fano variety (X, D) consists of a normal projective variety X and an effective $\mathbb{Q}$-divisor D such that $-(K_X + D)$ is ample $\mathbb{Q}$-Cartier divisor.
For example, $(X = \mathbb{P}^3, cS_4)$ for $c \in (0,1) \cap \mathbb{Q}$. If $D = 0$, log Fano = Fano.

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Definition (Fujita-Li)

A log Fano variety (X, D) is K-semistable if

$$\mathrm{FL}_{(X,D)}(E) := A_{(X,D)}(E) - S_{(X,D)}(E) \geq 0$$

for any prime divisor $E \subset Y \xrightarrow{\pi} X$. Here

$$A_{(X,D)}(E) := 1 + \mathrm{ord}_E(K_Y - \pi^*(K_X + D))$$

$$S_{(X,D)}(E) := \frac{1}{(-K_X - D)^n} \int_0^\infty vol(-\pi^*(K_X + D) - tE)dt$$

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- Recall Zariski decomposition on normal projective surface X: let D be pesudo-effective $\mathbb{Q}$-divisor, then there is a unique decomposition $D = P + N$ where $P, N \geq 0$ $\mathbb{Q}$-divisors such that $P.N_i = 0$ for each component of N, P is nef and the intersection matrix of components of N is negative or $N = 0$. In particular, $vol(D) = P^2$.

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- $-K_{Bl_p\mathbb{P}^2} - tE = \mu^*\mathcal{O}(3) - (t+1)E$ has Zariski decomposition $P_t = \mu^*\mathcal{O}(3) - (t+1)E$ for $0 \leq t \leq 2$.

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$$S_{Bl_p\mathbb{P}^2}(E) = \frac{1}{8}\int_0^2 (9 - (t+1)^2)dt = \frac{7}{6}$$

K-stability: how to check it ?

In general, how to check a given log Fano variety (X, D) is one of the challenging problem in K-stability theory.

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At present, it is an active research direction to check K-stablity of log Fano varieties. The main two approaches

- Equivariant criterion and Abban-Zhuang's adjunction of stability threshold.
- Moduli method.

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  2. Ascher-DeVleming-Liu (2021) proved K-moduli space of $(\mathbb{P}^1 \times \mathbb{P}^1, cC)$ is isomorphic to VGIT

$$\mathbb{P}\mathcal{E} /\!\!/_{L_t} PGL(4), \ L_t = \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) + p^*\mathcal{O}_{\mathbb{P}^9}(t)$$

where $p : \mathbb{P}\mathcal{E} \to |\mathcal{O}_{\mathbb{P}^3}(2)| = \mathbb{P}^9$ is a projective bundle parametrizing $(2, 4)$ complete intersections in $\mathbb{P}^3$.

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     where $p : \mathbb{P}\mathcal{E} \to |\mathcal{O}_{\mathbb{P}^3}(2)| = \mathbb{P}^9$ is a projective bundle parametrizing $(2,4)$ complete intersections in $\mathbb{P}^3$.
  3. Ascher-DeVleming-Liu (2022) gives full wall-crossing of K-moduli space for $(\mathbb{P}^3, cS_4)$, based on the work of Laza-O'Grady's work on moduli space of quartic K3 surfaces.

Equivariant criterion

Theorem (Zhuang)

Let G be an algebraic group acting on (X, D). Then (X, D) is K-semistable if and on if (X, D) is G-equivariant K-semistable.

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Theorem (Zhuang, Ilten-Suss)

Let (X, D) be a 2-dimensional log Fano with an effective $\mathbb{G}_m$-action λ. Then (X, D) is K-polystable if and only if the followings hold:

1. $\mathrm{FL}_{(X,D)}(F) > 0$ *for all vertical λ-invariant prime divisors F on X;*
2. $\mathrm{FL}_{(X,D)}(F) = 0$ *for all horizontal λ-invariant prime divisors F on X;*
3. $\mathrm{FL}_{(X,D)}(v) = 0$ *for the valuation v induced by the 1-PS λ.*

Example

$C = H_x + H_y + 4H_z \sim -2K_{Bl_p\mathbb{P}^2}$ where H_x the proper transform of the line $\{x = 0\} \subset \mathbb{P}^2$.

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Proof.

- $A_{(Bl_p\mathbb{P}^2,cC)}(H_z) = 1-4c \geq S_{(Bl_p\mathbb{P}^2,cC)}(H_z) = \frac{5}{6}(1-2c)$ implies $c \leq \frac{1}{14}$
- $A_{(Bl_p\mathbb{P}^2,cC)}(H_x) = 1-c \geq S_{(Bl_p\mathbb{P}^2,cC)}(H_x) = \frac{13}{12}(1-2c)$ implies $c \geq \frac{1}{14}$

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- The pair $(Bl_p\mathbb{P}^2, C)$ is toric. Computation of barycenters will show $(Bl_p\mathbb{P}^2, \frac{1}{14}C)$ is K-semistable. Or one can use a $\mathbb{G}_m$-equivariant criterion.

□

K-moduli spaces of log Fano varieties

- Due to many people's work (Jiang, Xu, Blum-Liu-Xu, Blum-Xu, Liu-Xu-Zhuang, Xu-Zhunag etc), there is a proper Artin stack of finite type $\mathfrak{P}^K(c)$ parametrizing K-semistable n-dimensional log Fano varieties (X, cD) with fixed volume $v = (-K_X)^n$ where $D \sim -2K_X$ and $c \in (0, \frac{1}{2}) \cap \mathbb{Q}$.

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- Moreover, $\mathfrak{P}^K(c)$ has good moduli space

$$\mathfrak{P}^K(c) \to \mathrm{P}^K(c)$$

in the sense of J. Alper, which locally looks like

$$[Spec(R)/G] \to Spec(R^G)$$

where G is a reductive algebraic group.

K-moduli wall-crossing

Theorem (Ascher-DeVleming-Liu- 2019)

There are finitely many rational numbers (i.e., walls) $0 < w_1 < \cdots < w_m < \frac{1}{2}$ *such that*

$$\overline{P}(c)^K \cong \overline{P}(c')^K \quad \text{for any } w_i < c, c' < w_{i+1} \text{ and } \text{ any } 1 \leq i \leq m-1.$$

Denote $\overline{P}^K(w_i, w_{i+1}) := \overline{P}^K(c)$ *for some* $c \in (w_i, w_{i+1})$*, then at each wall* w_i*, there is a flip (or divisorial contraction)*

$$\overline{P}^K(w_{i-1}, w_i) \longrightarrow \overline{P}^K(w_i) \longleftarrow \overline{P}^K(w_i, w_{i+1})$$

which fits into a local VGIT.

K-moduli of del pezzo pair of degree 8

- Let $\mathrm{P}^K(c)$ be the K-moduli space of 2-dimensional log Fano varieties with $(-K_X)^2 = 8$ and a general member is $(Bl_p\mathbb{P}^2, cC)$.

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- $C \in |-2K_{Bl_p\mathbb{P}^2}|$ can be viewed as $C = \pi^* D - 2E$ where $D \subset \mathbb{P}^2$

$$D = \{z^4 f_2(x, y) + z^3 f_3(x, y) + \cdots + f_6(x, y) = 0\}.$$

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$$D = \{z^4 f_2(x,y) + z^3 f_3(x,y) + \cdots + f_6(x,y) = 0\}.$$

Assume $f_2(x,y)$ has rank 2, then curve D has the form

$$az^4xy + z^3\widetilde{f_3}(x,y) + z^2 f_4(x,y) + zf_5(x,y) + f_6(x,y) = 0$$

Let $\mathbb{P}V \cong \mathbb{P}^{20}$ be the parameter space of such D and there is $T = (\mathbb{C}^*)^2$-action on $\mathbb{P}V$ and define GIT space $\mathbb{P}V /\!\!/ T$.

Moduli space

- Let $X = X_C \to Bl_p\mathbb{P}^2$ be the double cover branched along smooth curve $C \sim -2K_{Bl_p\mathbb{P}^2}$, then X is a K3 surface with anti-symplectic involution $\tau : X \to X$. Then $NS(X)$ contains

$$\begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}.$$

Its period domain $\mathcal{D}$ is determined transcendental lattice $U^2 \oplus E_8 \oplus E_7 \oplus A_1$.

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- Via a period point of K3 surfaces, there is biratonal map

$$\mathrm{P}^K(c) \dashrightarrow \mathcal{F} = \Gamma \setminus \mathcal{D}, \; [(Bl_p\mathbb{P}^2, C)] \mapsto H^{2,0}(S_C) \mod \Gamma$$

  if $\mathrm{P}^K(c)$ is nonempty.

Two divisors $\mathcal{F}$

- Hyperelliptic divisor H_h on $\mathcal{F}$: $X \xrightarrow{2:1} Bl_p\mathbb{P}^2$ branched along a general curve $C \in |-2K_{Bl_p\mathbb{P}^2}|$ tangent the (-1)-curve E.

$$NS(X) = \left(\begin{array}{c|ccc} & L & E_1 & E_2 \\ \hline L & 2 & 0 & 0 \\ E_1 & 0 & -2 & 1 \\ E_2 & 0 & 1 & -2 \end{array}\right)$$

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- Unigonal divisor H_u on $\mathcal{F}$: $X \xrightarrow{2:1} Bl_p\widetilde{\mathbb{P}(1,1,4)} \to Bl_p\mathbb{P}(1,1,4)$.

$$NS(X) = \left(\begin{array}{c|ccc} & E' & F' & H'_y \\ \hline E' & -2 & 0 & 2 \\ F' & 0 & -2 & 1 \\ H'_y & 2 & 1 & -2 \end{array}\right)$$

Main results 1

Theorem A (Pan-Si-Wu,2023)

1. *The walls for K-moduli space* $\mathrm{P}^K(c)$ *are*

$$W_h = \{ \frac{1}{14}, \frac{5}{58}, \frac{1}{10}, \frac{7}{62}, \frac{1}{8}, \frac{5}{34}, \frac{1}{6}, \frac{7}{38}, \frac{1}{5}, \frac{5}{22}, \frac{2}{7} \}$$

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$$W_u = \left\{ \frac{29}{106}, \frac{31}{110}, \frac{2}{7}, \frac{35}{118} \right\}$$

2. *If* $c \in (0, \frac{1}{14})$, $\mathrm{P}^K(c)$ *is empty. If* $c \in (\frac{1}{14}, \frac{5}{58})$, $\mathrm{P}^K(c) \cong \mathbb{P}V /\!\!/ T$.

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3. *There are two divisorial contraction morphisms* $\mathrm{P}^K(w+\epsilon) \to \mathrm{P}^K(w)$ *at* $w = \frac{5}{58}$ *and* $w = \frac{29}{106}$. *The exceptional divisors* $E_w^+ \subset \mathrm{P}^K(w+\epsilon)$ *is birational to hyperelliptic divisor* H_h*(resp. unigonal divisor* H_u *).*

Table for K-wall

wall	curve B on $\mathbb{P}^2$	weight	curve singularity at p
$\frac{1}{14}$	$x^4zy = 0$	(1,0,0)	A_1
$\frac{5}{58}$	$x^4z^2 + x^3y^3 = 0$	(0,2,3)	A_2
$\frac{1}{10}$	$x^4z^2 + x^3zy^2 + a \cdot x^2y^4 = 0$	(0,1,2)	A_3
$\frac{7}{62}$	$x^4z^2 + xy^5 = 0$	(0,2,5)	A_4
$\frac{1}{8}$	$x^4z^2 + x^2zy^3 + a \cdot y^6 = 0,$	(0,1,3)	A_5 tangent to L_z
	$x^3f_3(z, y) = 0$	(0,1,1)	D_4
$\frac{5}{34}$	$x^4z^2 + xzy^4 = 0$	(0,1,4)	A_7 with a line
	$x^3z^2y + x^2y^4 = 0$	(0,2,3)	D_5
$\frac{1}{6}$	$x^4z^2 + zy^5 = 0$	(0,1,5)	A_9 with a line
	$x^3z^2y + x^2zy^3 + a \cdot xy^5 = 0$	(0,1,2)	D_6

Table: K-moduli walls from Gorenstein del Pezzo $\mathbb{F}_1 = Bl_{[1,0,0]}\mathbb{P}^2$

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	$x^3z^3 + x^2y^4 = 0$	(0,3,4)	E_6
$\frac{1}{5}$	$x^3z^2y + xzy^4 = 0$	(0,1,3)	D_8 with L_z
$\frac{5}{22}$	$x^3z^2y + zy^5 = 0$	(0,1,4)	D_{10} with L_z
	$x^3z^3 + x^2zy^3 = 0$	(0,2,3)	E_7
$\frac{2}{7}$	$x^3z^3 + xy^5 = 0$	(0,3,5)	E_8

Table: K-moduli walls from Gorenstein del Pezzo $\mathbb{F}_1 = Bl_{[1,0,0]}\mathbb{P}^2$

Table for K-walls

wall	curve B on $\mathbb{P}^2$	weight	curve singularity at p
$\frac{7}{38}$	$x^3z^2y + y^6 = 0$	(0,2,5)	D_7 tangent to L_z
	$x^3z^3 + x^2y^4 = 0$	(0,3,4)	E_6
$\frac{1}{5}$	$x^3z^2y + xzy^4 = 0$	(0,1,3)	D_8 with L_z
$\frac{5}{22}$	$x^3z^2y + zy^5 = 0$	(0,1,4)	D_{10} with L_z
	$x^3z^3 + x^2zy^3 = 0$	(0,2,3)	E_7
$\frac{2}{7}$	$x^3z^3 + xy^5 = 0$	(0,3,5)	E_8

Table: K-moduli walls from Gorenstein del Pezzo $\mathbb{F}_1 = Bl_{[1,0,0]}\mathbb{P}^2$

wall	curve B on $\mathbb{P}(1,1,4)$	weight	(a,b,m)
$\frac{29}{106}$	$z^3 + z^2x^4 = 0$	(1,0,4)	$(0,1,0)$
$\frac{31}{110}$	$z^3 + zyx^7 = 0$	(2,0,7)	$(1,1,1)$
$\frac{2}{7}$	$z^3 + y^2x^{10} = 0$	(3,0,10)	$(2,1,2)$
$\frac{35}{118}$	$z^3 + zy^2x^6 + y^3x^9 = 0$	(1,0,3)	(1,0,1)

Table: K-moduli walls from index 2 del Pezzo $Bl_{[1,0,0]}\mathbb{P}(1,1,4)$

Main results 2

Define the Hasset-Keel-Looijenga (HKL) model for $\mathcal{F}$

$$\mathcal{F}(s) := \mathrm{Proj}\big(\bigoplus_m H^0(\mathcal{F}, m(\lambda + sH_h + 25sH_u)\big)$$

By Baily-Borel's work, $\mathcal{F}(0) = \mathcal{F}^*$ is Baily-Borel's compactification for $\mathcal{F}$ with boundaries $\mathcal{F}^* - \mathcal{F}$ consisting of modular curves.

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Theorem B (Pan-Si-Wu,2023)

There is natural isomorphism $\mathrm{P}^K(c) \cong \mathcal{F}(s)$induced by the period map under the transformation

$$s = s(c) = \frac{1-2c}{56c-4}$$

where $\frac{1}{14} < c < \frac{1}{2}$. In particular, $\mathrm{P}^K(c)$ will interpolates the GIT space $\mathbb{P}V /\!\!/ T$ and Baily-Borel compactification $\mathcal{F}^*$. The walls are $w = \frac{1}{n}$ and

$$n \in \{1, 2, 3, 4, 6, 8, 10, 12, 16, 25, 27, 28, 31\ \}$$

Sketch of proof of Theorem A

- Step1: To determine K-semistable degeneration. By using some classification results of index ≤ 2 del pezzo surface and normalised volume comparison due to Chi Li, Li-Liu, we can show each $(X, cC) \in \mathrm{P}^K(c)$, then X is either $Bl_p\mathbb{P}^2$ or $Bl_p\mathbb{P}(1,1,4)$.

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- Step2: Local VGIT structure of K-moduli implies if $(X, cC) \in P^K(w)$ admits 1-PS λ and thus $\mathrm{FL}(E_\lambda) = 0$ where E_λ is exceptional divisor of certain weighted blowup determined by λ. e.g, $X = Bl_{[1,0,0]}\mathbb{P}^2$ and for some $\lambda = [0, m_1, m_2]$ on X,

$$A_{(X,cC)}(E_\lambda) = a + b - mc, \quad S_{(X,cC)}(E_\lambda) = \frac{14a + 13b}{12}(1 - 2c)$$

Then $A_{(X,cC)}(E_\lambda) = S_{(X,cC)}(E_\lambda)$ will all potential walls.

- Step3: To determine the 1st walls and then keep track of wall crossing at all centers for each walls.

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  Following the arguments of Liu-Xu, show for $\frac{1}{14} < c < \frac{1}{14} + \epsilon$ and any K-degeneration (X_0, cC_0) of $(Bl_p\mathbb{P}^2, cC)$, X_0 is still $Bl_p\mathbb{P}^2$, then

$$\mathfrak{P}^K \hookrightarrow \mathbb{P}V.$$

  Then explicit wall-crossing are followed by analysis of local VGIT at each wall $w \in W_u \cup W_h$.

Some remarks:

- The explicit wall-crossing from $\mathbb{P}V /\!\!/ T$ to $\mathcal{F}^*$ will be useful to calculate the topological invariants and intersection theory on the moduli space $\mathcal{F}$.

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- For higher dimensional log Fano pairs, to find walls of their K-moduli seems much harder than dimension 2. The arithmetic stratifications should be powerful to predict walls for K-moduli of log Fanos related to K3 surfaces (even irreducible holomorphic symplectic varieties).

Some remarks:

- The explicit wall-crossing from $\mathbb{P}V /\!\!/ T$ to $\mathcal{F}^*$ will be useful to calculate the topological invariants and intersection theory on the moduli space $\mathcal{F}$.
- For higher dimensional log Fano pairs, to find walls of their K-moduli seems much harder than dimension 2. The arithmetic stratifications should be powerful to predict walls for K-moduli of log Fanos related to K3 surfaces (even irreducible holomorphic symplectic varieties).
- It should be interesting to look at the behavior of $c > \frac{1}{2}$ and $c = \frac{1}{2}$. For $c > \frac{1}{2}$, by Alexeev-Engel and Alexeev-Engel-Han's work, the KSBA moduli space compactifying pairs $(Bl_p\mathbb{P}^2, cC)$ and their slc degeneration has a natural normalization— Toroidal compactification of $\mathcal{F}$.
  For $c = \frac{1}{2}$, it is expected to have a moduli theory for log CY to connect wall crossing from K-moduli to KSBA moduli.

Thank you for attention !